



PLACE LABEL HERE

Tasmanian Certificate of Education

MATHEMATICS - SPECIALISED

Senior Secondary 5C

Subject Code: MSP5C

External Assessment – AMENDED PAPER

2004

Time: 3 hours

On the basis of your performance in this examination, the examiners will provide results on the following criteria taken from the syllabus statement:

- Criterion 6** Demonstrate an understanding of finite and infinite sequences and series.
- Criterion 7** Demonstrate an understanding of matrices and linear transformations.
- Criterion 8** Use differential calculus in the study of functions.
- Criterion 9** Use integral calculus in the study of functions.
- Criterion 10** Demonstrate an understanding of complex numbers.

Pages: 11
Questions: 24

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CANDIDATE INSTRUCTIONS

Candidates **MUST** ensure that they have addressed **ALL** of the externally assessed criteria on this examination paper.

This examination paper has **six sections**. Candidates must answer **ALL** questions.

It is suggested that candidates spend approximately 20 minutes on Sections A – E and approximately 80 minutes on Section F.

The 2004 Mathematics – Specialised Formula Sheet can be used throughout the examination.

No other printed material is allowed into the examination.

The presentation of answers and the statement of arguments leading to answers will be considered when determining your result on each criterion.

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching and an approved scientific and/or graphics calculator (memory may be retained).

Answer each **section** in a separate answer booklet.

SECTION A – SEQUENCES AND SERIES

This section assesses **criterion 6**.

Use a separate answer booklet for this section.

Question 1

Find the n^{th} term of the series whose sum to n terms, S_n , is given by $S_n = \frac{3^{n+1} - 3}{2}$. (3 marks)

Question 2

Prove that the sequence $\left\{2 - \frac{5}{n}\right\}$ converges to 2. (3 marks)

Question 3

Given $f(x) = e^{-2x}$, use the MacLaurin series expansion to find the first four terms and the general term in x^n . (5 marks)

Question 4

Using the method of differences, show that $\sum_{r=1}^n \frac{1}{(3r-2)(3r+1)} = \frac{n}{3n+1}$ and hence evaluate

$\sum_{r=1}^{\infty} \frac{1}{(3r-2)(3r+1)}$. (7 marks)

SECTION B – MATRICES & LINEAR TRANSFORMATIONS

This section assesses **criterion 7**.

Use a separate answer booklet for this section.

Question 5

The circle $x^2 + y^2 = 1$ is transformed by a shear of factor 2 parallel to the y -axis. Write the transformation matrix and find the area of the image of the circle. (3 marks)

Question 6

Find the 2×2 transformation matrix which transforms $(1, 1)$ to $(2, 2)$ and $(1, 2)$ to $(4, 3)$. (3 marks)

Question 7

Given $A = \begin{pmatrix} 2 & -5 \\ 3 & 1 \end{pmatrix}$, find scalar values a , b and c (not all zero) for which $aI + bA + cA^2 = \mathbf{0}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. (5 marks)

Question 8

The parabola $y = x^2 + 2x + 3$ is rotated clockwise through 90° about the origin and then reflected in the line $y = x$. Find the equation of the image of the parabola under this composition of transformations. (7 marks)

SECTION C – DIFFERENTIAL CALCULUS
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This section assesses **criterion 8**.

Use a separate answer booklet for this section.

Question 9

If $e^x + e^y = 1$, find $\frac{dy}{dx}$ in simple form. (3 marks)

Question 10

Find the equation of the normal to the curve $y = \frac{x}{\sin x}$ at the point where $x = \frac{\pi}{2}$. (3 marks)

Question 11

If $y = e^x(\cos x + \sin x)$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. (5 marks)

Question 12

Differentiate $\arcsin \sqrt{1-x^2}$, for $0 \leq x \leq 1$.

Hence, or otherwise, show that $\arcsin x + \arcsin \sqrt{1-x^2} = \frac{\pi}{2}$, for $0 \leq x \leq 1$. (7 marks)

SECTION D – INTEGRAL CALCULUS

This section assesses **criterion 9**.

Use a separate answer booklet for this section.

Question 13

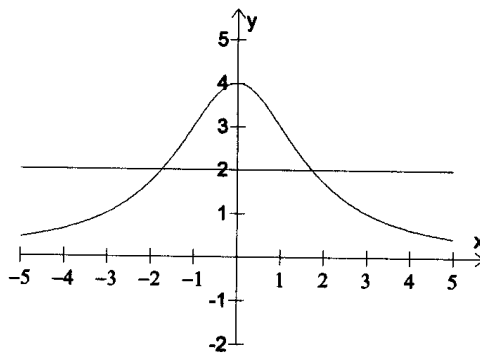
Find the indefinite integral $\int \sin^2 3x dx$.

(3 marks)

Question 14

In the diagram, the area bounded by the curve $y = \frac{12}{3+x^2}$ and the line $y = 2$ is rotated about the y -axis. Find the exact value of the volume of the solid so generated.

(3 marks)


Question 15

Find a general solution to the differential equation $\frac{dy}{dx} = \frac{y^2 - 1}{2}$.

(5 marks)

Question 16

Use the substitution $u = x + \frac{1}{x}$ to find the exact value of $\int_1^2 \frac{x^2 - 1}{x^4 + 3x^2 + 1} dx$.

(7 marks)

SECTION E – COMPLEX NUMBERS

This section assesses **criterion 10**.

Use a separate answer booklet for this section.

Question 17

$$z = 5 + 2i \text{ and } w = -2 - 6i.$$

Calculate the value of $z + w$ and $z - w$, and illustrate all four complex numbers on an Argand diagram. (3 marks)

Question 18

If $z = a + bi$ and $w = c + di$, prove that $\bar{z} + \bar{w} = \overline{z + w}$. (3 marks)

Question 19

Find all pairs of integers x and y that satisfy $(x + iy)^2 = 24 + 10i$. (5 marks)

Question 20

Solve for z if $\frac{2z}{2+i} + 3 - 2i = (1-i)z$. (7 marks)

SECTION F

Use a separate answer booklet for this section.

Question 21

This question assesses **criteria 6, 7 and 10**.

It is recommended that you spend approximately 30 minutes on this question.

- (a) Use the formula for the sum of a geometric series to show that

$$\sum_{k=1}^n (z + z^2 + z^3 + \dots + z^k) = \frac{nz}{1-z} - \frac{z^2(1-z^n)}{(1-z)^2}, \quad z \neq 1. \quad (7 \text{ marks})$$

- (b) If $z = \cos 2\theta + i \sin 2\theta$, show that $\frac{z}{1-z} = \frac{i}{2 \sin \theta} (\cos \theta + i \sin \theta)$. (7 marks)

- (c) As before, $z = \cos 2\theta + i \sin 2\theta$. By considering the imaginary parts of the result in part (a),

$$\text{Im} \left\{ \sum_{k=1}^n (z + z^2 + z^3 + \dots + z^k) \right\} = \text{Im} \left\{ \frac{nz}{1-z} - \frac{z^2(1-z^n)}{(1-z)^2} \right\}, \quad z \neq 1,$$

and the result from part (b), show that

$$\sum_{k=1}^n (\sin(2\theta) + \sin(2 \cdot 2\theta) + \sin(3 \cdot 2\theta) + \dots + \sin(k \cdot 2\theta)) = \frac{(n+1) \sin 2\theta - \sin(n+1)2\theta}{4 \sin^2 \theta}. \quad (8 \text{ marks})$$

Question 22

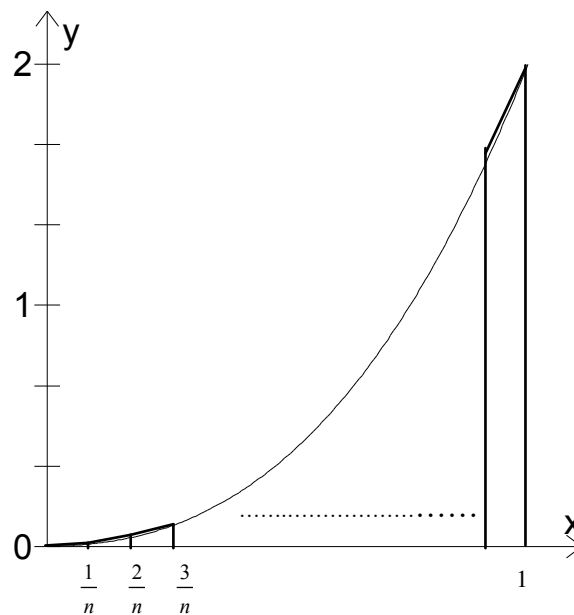
This question assesses **criteria 6 and 9**.

It is recommended that you spend approximately 13 minutes on this question.

The area under the curve $y = x^3 + x^2$ between the ordinates at $x = 0$ and $x = 1$ is to be approximated by n trapezoidal strips of the same width.

(a) Use the trapezoidal rule to show that

$$\int_0^1 (x^3 + x^2) dx \approx \frac{7n^2 + 5}{12n^2}, \text{ where } n \text{ is the number of the trapezoidal strips taken.} \quad (7 \text{ marks})$$



(b) Verify, using calculus methods, that $\int_0^1 (x^3 + x^2) dx = \lim_{n \rightarrow \infty} \frac{7n^2 + 5}{12n^2}$. (3 marks)

Question 23

This question assesses **criteria 8 and 9**.

It is recommended that you spend approximately 25 minutes on this question.

(a) Show that $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x + 3\tan^4 x$, and hence show that $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx = \frac{\pi}{4} - \frac{2}{3}$. (6 marks)

(b) Consider $y = 5xe^{5x}$.

(i) Find, and determine the nature of, any stationary points as well as any points of inflection of $y = 5xe^{5x}$. (6 marks)

(ii) Sketch the graph of $y = 5xe^{5x}$ showing any essential features. (3 marks)

(iii) Find the exact value of the area between the graph of $y = 5xe^{5x}$, the x -axis and the ordinates at $x = -1$ and $x = 0$. (4 marks)

Question 24

This question assesses **criterion 7**.

It is recommended that you spend approximately 12 minutes on this question.

(a) Find x if $Y^{-1} = -Y$, and $Y = \begin{pmatrix} 2x & 1 \\ 2x-1 & 1 \end{pmatrix}$. (2 marks)

(b) Using Gauss-Jordan elimination, find $k \neq 0$ such that the equations

$$\begin{aligned}x + y + kz &= 0 \\3x - 7y + z &= 0 \\kx - 8y - z &= 0\end{aligned}$$

have solutions other than $x = y = z = 0$. (7 marks)